Radiation

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致谢：本课件中部分资料来自李成才老师（特别是关于辐射的部分）。

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Outline

- Introduction
- Concepts
- Absorption
- Scattering
- Radiative transfer
- Radiative equilibrium temperature
- Radiative heating and cooling

思考题

- 问题一：说明不对称因子 g 的意义： $g > 0, g < 0, g = 0$ 。假设散射时向上和向下的散射光强分别是一样的（semi-isotropic），在这种情况下，g 可能达到的最大值是多少？g 可能达到1么？$g = 1$ 和 $g = -1$ 分别意味着什么？

- 问题二：‘蓝月亮’是指月亮呈现蓝色，是很罕见的现象。思考肉眼从地面向上看到‘蓝月亮’所需条件，以及为什么这种现象很少出现。

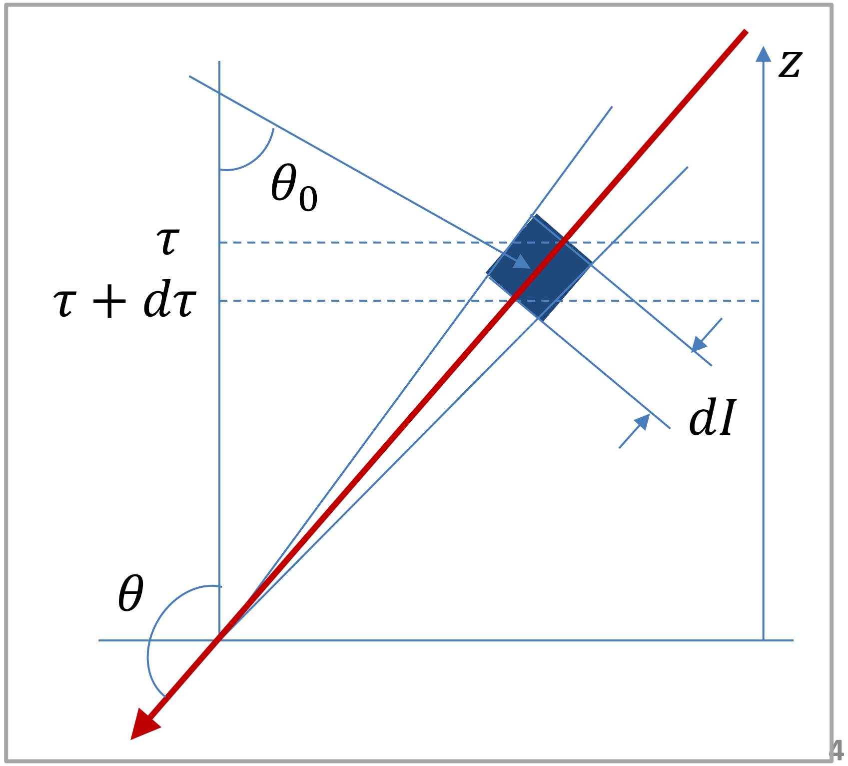
https://zhuanlan.zhihu.com/p/33472940
https://m.sohu.com/n/464038572/?_f=m-article_ch

Radiative Transfer 辐射传输

- 当我们观察来自天空某一方向的某一波长的辐亮度大小，实际上是接收在对应立体角中自眼睛直到大气上界整个气柱所发来的光。
- 让我们分析经过一段距离 dl，气柱辐亮度的变化 dI。

假设平面平行大气

注意 θ_0 和 θ 在定义上的区别



辐射传输

- 某一波长的辐亮度 I 在经过一段气柱后的变化由下列四种因素引起：
  1、由于这段气柱的吸收和一次散射，I 经过这段气柱后受到衰减；
  2、由于太阳光直射到这段气柱上，气柱在该方向发出散射光，即一次散射
  3、气柱周围来自各个方向的散射光射到这段气柱上再发生散射，即多次散射；
  4、这段气柱中大气的热辐射。

Bill定律：吸收和一次散射导致的消光作用

- Bill定律： e指数衰减

$$dI = -I \cdot k_{ex} \cdot dl$$

若体积元内粒子相同：

$$dI = -I \cdot N\sigma_{ex} \cdot dl$$

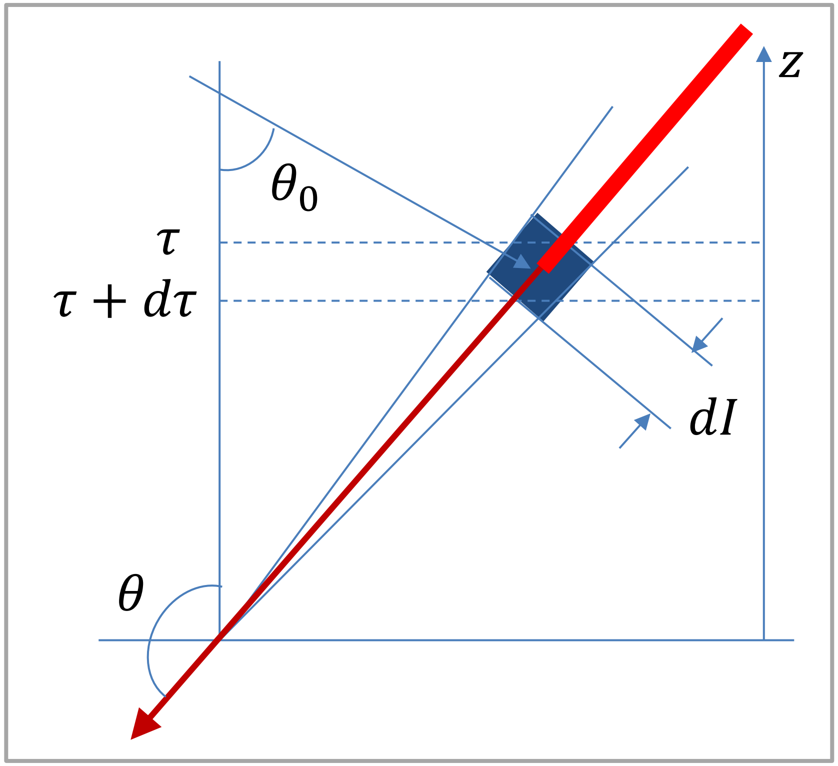
$$= -I \cdot (N\sigma_{ab} + N\sigma_{sc}) \cdot dl$$

k_{ex}为体积消光系数　N为粒子的数密度

σ_{ex}为单个粒子的消光截面

这里省略波长、高度、角度等标识

注意 θ_0 和 θ 在定义上的区别



（太阳辐射）一次散射

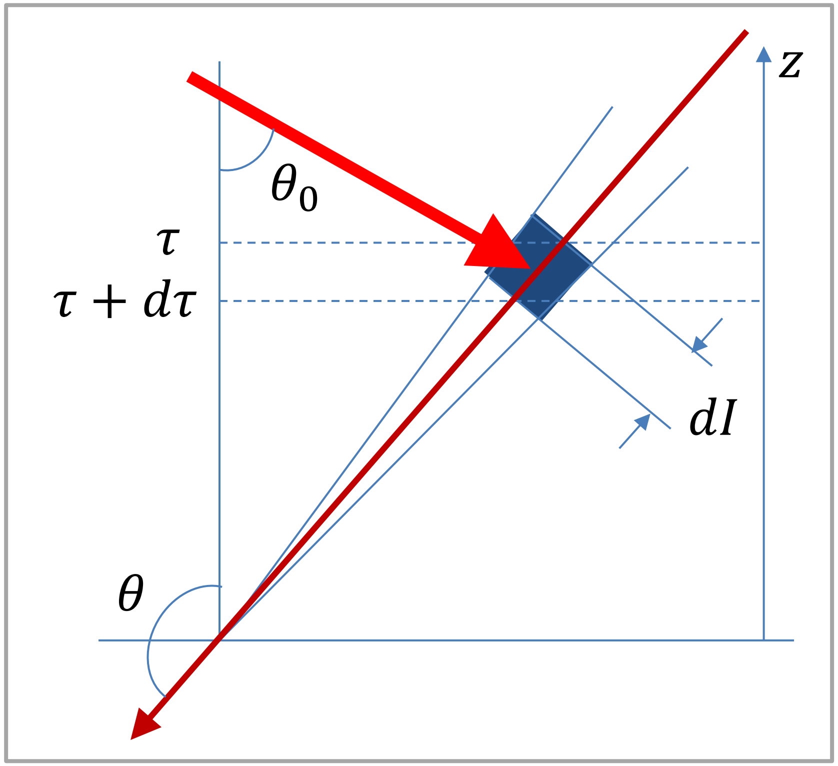
$$dF_s(\theta,\varphi)=\frac{F_{\boxast}}{R^2}\cdot\beta(\theta,\varphi,\theta_0,\varphi_0)\cdot dv$$

$$dI_s(\theta,\varphi)\cdot\Delta\Omega$$ 为什么没有cos项？

$$=\frac{F_{\boxast}}{R^2}\cdot\beta(\theta,\varphi,\theta_0,\varphi_0)\cdot\Delta S\cdot dl$$

$$=F_{\boxast}\cdot\beta(\theta,\varphi,\theta_0,\varphi_0)\cdot\Delta\Omega\cdot dl$$

$$dI_s(\theta,\varphi)=F_{\boxast}\cdot\beta(\theta,\varphi,\theta_0,\varphi_0)\cdot dl$$



$\beta(\theta,\varphi,\theta_0,\varphi_0)$为散射方向性函数

对于太阳辐射的一次散射：$F_{\boxast}=F_0e^{-\tau(z)/\cos\theta_0}$

F_0 为TOA入射太阳辐射

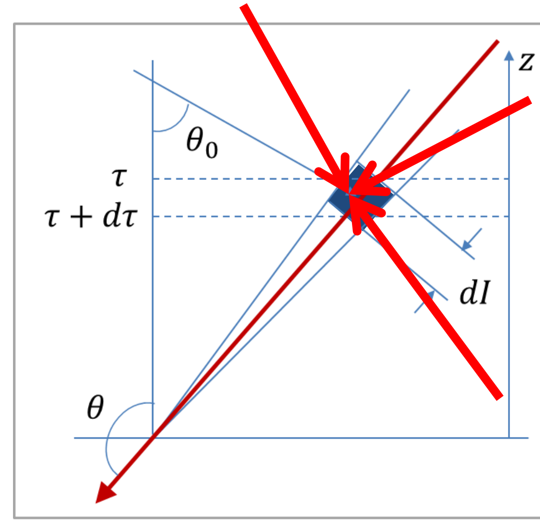
这里省略波长和高度标识，假设平面平行大气

注意 θ_0 和 θ 在定义上的区别

多次散射

- 单次散射

$$dI_s = F_{\boxtimes} \cdot \beta(\theta, \varphi, \theta_0, \varphi_0) \cdot dl$$



- 多次散射：

入射光为来自四面八方的散射光：

$$dF' = I(\theta', \varphi') \cdot d\Omega'$$

照射到体积元的辐射；为什么没有cos项？

$$dI_s = \int_{4\pi} \beta(\theta, \varphi, \theta', \varphi') \cdot dF' \cdot dl$$

$$= \int_0^{2\pi} \int_0^{\pi} I(\theta', \varphi') \cdot \beta(\theta, \varphi, \theta', \varphi') \cdot \sin\theta' \cdot d\theta' d\varphi' dl$$

这里省略波长和高度标识，假设平面平行大气

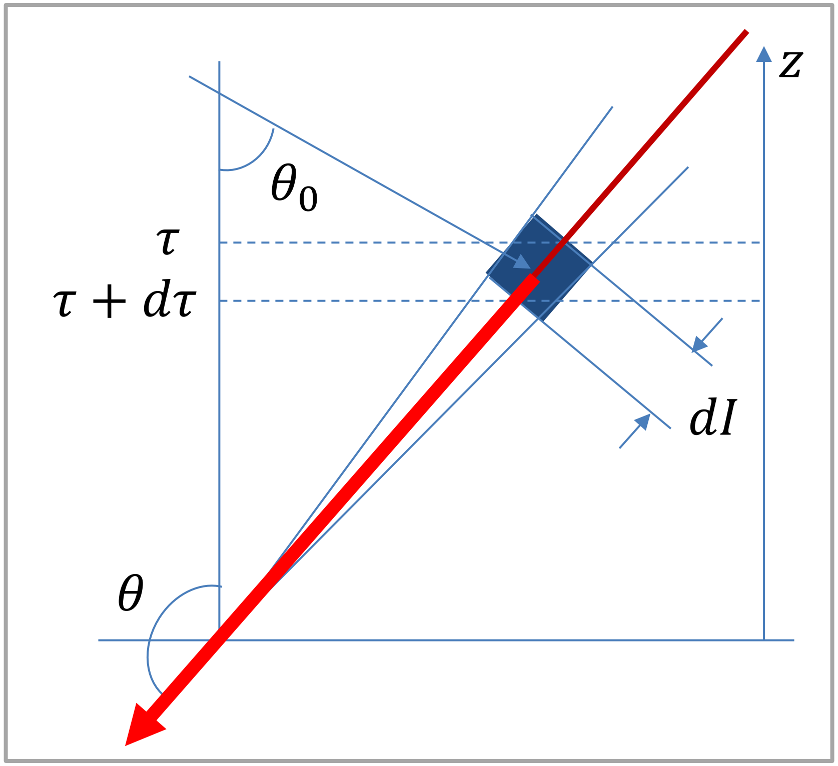
热辐射

➢ 气柱发出的热辐射：
由普朗克定律
和基尔霍夫定律确定

$$dI = B(T) \cdot \varepsilon$$

$$= B(T) \cdot k_{ab} \cdot dl$$

发射率 = 吸收率



这里省略波长和高度标识，假设平面平行大气

平面平行大气中的辐射传输方程

- 综合考虑衰减、一次散射、多次散射、热辐射：

$$dI = -I(z,\theta,\varphi)\cdot k_{ex}(z)\cdot dl$$

$$+F_0 e^{-\tau(z)/\cos\theta_0}\cdot\beta(z,\theta,\varphi,\theta_0,\varphi_0)\cdot dl$$

$$+\int_0^{2\pi}\int_0^{\pi} I(z,\theta',\varphi')\cdot\beta(z,\theta,\varphi,\theta',\varphi')\cdot\sin\theta'\cdot d\theta' d\varphi' dl$$

$$+B\big(T(z)\big)\cdot k_{ab}(z)\cdot dl$$

这里省略波长标识，假设平面平行大气

平面平行大气中的辐射传输方程

- 平面平行大气： $dl = dz/\cos\theta = dz/\mu$

  令： $\mu = \cos\theta$ $\mu_0 = \cos\theta_0$

  相函数： $P(\theta,\varphi) = 4\pi\beta(\theta,\varphi)/k_{sc}$

  单次散射反照率： $\omega_0 = k_{sc}/k_{ex}$

- 用光学厚度 τ 坐标代替 z 坐标：

$$d\tau = -k_{ex}dz = -\mu \cdot k_{ex}dl$$

τ 坐标从TOA算起，向下增加，到地面为τ_1 【整层大气垂直光学厚度】

这里省略波长标识

注意关于太阳天顶角 θ_0 的习惯定义

平面平行大气中的辐射传输方程

- Schwarzschild 史瓦西传输方程（变形形式），J 为源函数

$$\mu \frac{dI}{d\tau} = I - J$$

向上 (+)：$\mu > 0, d\tau < 0$
向下 (-)：$\mu < 0, d\tau > 0$

$$J = \frac{\omega_0}{4\pi}\left\{F_0 e^{-\tau/\mu_0} \cdot P(\tau,\theta,\mu,\theta_0,\varphi_0) + \int_0^{2\pi}\int_{-1}^{1} I(\tau,\theta',\varphi') \cdot P(\tau,\theta,\varphi,\theta',\varphi') \cdot d\mu' d\varphi' \right\}$$

$$+(1-\omega_0)B\big(T(\tau)\big)$$

注意： $d\tau = -\mu k_{ex} dl$　　$k_{sc} = \omega_0 k_{ex}$　　$k_{ab} = (1-\omega_0)k_{ex}$

ω_0 为单次散射反照率

平面平行大气中的辐射传输方程

- Θ是从太阳直射光方向（$\pi-\theta_0$，$\pi+\varphi_0$）到散射光方向（θ,φ）之间的夹角

$$\cos\Theta = \cos\theta\cos(\pi-\theta_0) + \sin\theta\sin(\pi-\theta_0)\cos[\varphi-(\pi+\varphi_0)]$$

$$= -\cos\theta\cos\theta_0 - \sin\theta\sin\theta_0\cos(\varphi-\varphi_0)$$

$$= -\mu\mu_0 - \sqrt{1-\mu^2}\sqrt{1-\mu_0^2}\cos(\varphi-\varphi_0)$$

注意关于太阳天顶角 θ_0 和太阳方位角 ϕ_0 的习惯定义

- 可见，散射与θ、θ_0、$\varphi-\varphi_0$有关

平面平行大气中的辐射传输方程

- 在不同的问题中，源函数 J 可以作相应简化：
  - 对短波辐射，热辐射项可以不考虑
  - 对晴空、清洁条件下的红外辐射，散射项可以不考虑
  - 讨论红外辐射在云中传输这类问题时，则一次散射项有时可以不计
- 平面平行大气假设：在很多情况下适用；但在天空有不均匀分布的云、研究曙暮光等情况时，可能需要考虑三维辐射传输以获得更准确结果。

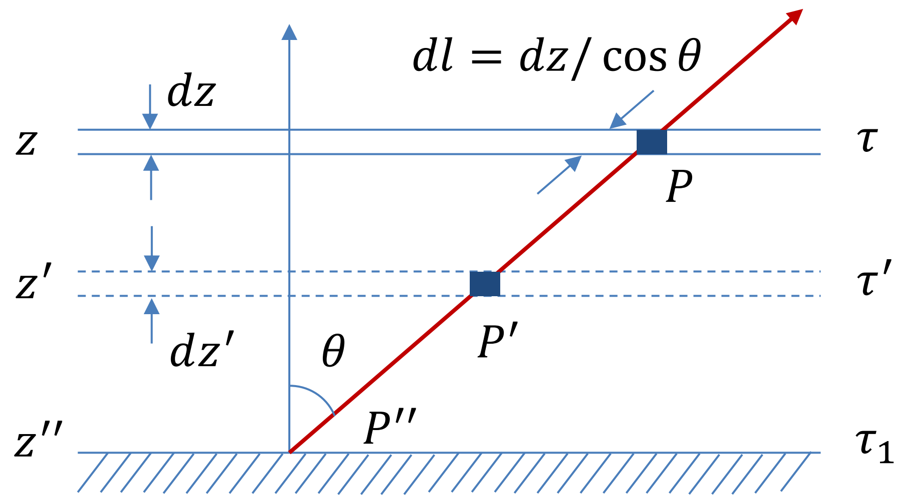
Solution to Radiative Transfer: Thermal Radiation

- Integral equation for thermal radiation:
  - No scattering of thermal radiation
  - No solar radiation

$$J = B\left(T(\tau)\right)$$

$$\mu \frac{dI}{d\tau} = I - J$$

$$= I - B\left(T(\tau)\right)$$



Modified from Richard Goody book

We get:

$$I = I_b \cdot e^{-(\tau_b - \tau)/\mu} + \frac{1}{\mu} \cdot \int_{\tau}^{\tau_b} B\left(T(\tau')\right) \cdot e^{-(\tau' - \tau)/\mu} \cdot d\tau'$$

Contribution of boundary

Contribution of atmosphere

Upward and Downward Radiance

- Distinguish upward (I^+) and downward (I^-) radiance:

  Upper and lower boundary conditions:

$$I^-(\tau = 0) = 0 \qquad \text{for } -1 \le \mu \le 0$$

$$I^+(\tau = \tau_1) = B\left(T_g\right) \qquad \text{for } 0 \le \mu \le +1; \text{ g is ground}$$

- Therefore:

$$I^+ = B\left(T_g\right) \cdot e^{-(\tau_1 - \tau)/\mu} + \frac{1}{\mu} \cdot \int_{\tau}^{\tau_1} B\left(T(\tau')\right) \cdot e^{-(\tau' - \tau)/\mu} \cdot d\tau'$$

$$I^- = -\frac{1}{\mu} \cdot \int_{0}^{\tau} B\left(T(\tau')\right) \cdot e^{-(\tau' - \tau)/\mu} \cdot d\tau'$$

I^+ and I^- are always positive values

Flux Density

$$F = \int_{4\pi} I \cdot \mu \cdot d\Omega \quad = \int_{-1}^{+1} \int_{0}^{2\pi} I \cdot \mu \cdot d\varphi \cdot d\mu$$

【 stratified 水平均一】 $= \int_{-1}^{+1} 2\pi \cdot I \cdot \mu \cdot d\mu$

$$= F^{+} - F^{-}$$

$$F^{+} = \int_{0}^{+1} 2\pi \cdot I^{+} \cdot \mu \cdot d\mu$$

Upward

$$F^{-} = \int_{0}^{-1} 2\pi \cdot I^{-} \cdot \mu \cdot d\mu$$

Downward

- F : (Net) flux density. Positive or negative value
- F^{+} : Upward flux density. Positive value
- F^{-} : Downward flux density. Positive value

Flux Density

$$F^{+} = \int_{0}^{+1} 2\pi \cdot I^{+} \cdot \mu \cdot d\mu$$

$$= 2\pi \cdot B(T_g) \cdot E_3(\tau_1 - \tau) + 2\pi \cdot \int_{E_3(\tau_1-\tau)}^{1/2} B(T(\tau')) \cdot dE_3(\tau' - \tau)$$

$$F^{-} = \int_{0}^{-1} 2\pi \cdot I^{-} \cdot \mu \cdot d\mu = 2\pi \cdot \int_{E_3(\tau)}^{1/2} B(T(\tau')) \cdot dE_3(\tau' - \tau)$$

$$F = F^{+} - F^{-}$$

Third exponential integral: $E_3(x) = \int_0^1 y \cdot e^{-x/y} dy = \int_1^{\infty} \frac{e^{-xt}}{t^3} dt$

$$I^{+} = B(T_g) \cdot e^{-(\tau_1-\tau)/\mu} + \frac{1}{\mu} \cdot \int_{\tau}^{\tau_1} B(T(\tau')) \cdot e^{-(\tau'-\tau)/\mu} \cdot d\tau'$$

$$I^{-} = -\frac{1}{\mu} \cdot \int_{0}^{\tau} B(T(\tau')) \cdot e^{-(\tau'-\tau)/\mu} \cdot d\tau'$$

Radiance to Space Approximation: *Radiative Heating*

For thermal radiation only, with no solar radiation & scattering

- Characteristics of atmosphere:
  - ✓ The upper edge layer looses energy to the outer space; and the middle layers receive and loose radiation with certain portion lost by net radiation towards the outer space
  - ✓ Air temperature in lower and mid atmospheres are normally within 40 K (16%) of the vertical average
- To calculating radiative heating in a layer of interest, assuming all layers and the ground have the same temperature as that level, so that all radiation exchange, except with outer space, is identically zero. Therefore:

$$I^{+} = B(T_g) \cdot e^{-(\tau_1-\tau)/\mu} + \frac{1}{\mu} \cdot \int_{\tau}^{\tau_1} B(T(\tau')) \cdot e^{-(\tau'-\tau)/\mu} \cdot d\tau' = B(T(\tau)) \quad \text{for } 0 \le \mu \le +1$$

$$I^{-} = -\frac{1}{\mu} \cdot \int_{0}^{\tau} B(T(\tau')) \cdot e^{-(\tau'-\tau)/\mu} \cdot d\tau' = B(T(\tau)) \cdot (1 - e^{\tau/\mu}) \quad \text{for } -1 \le \mu \le 0$$

Radiance to Space Approximation

Thermal radiation heating: $\Delta S \cdot dz \cdot \rho \cdot \delta q = -dF \cdot \Delta S \cdot \delta t$

Thus, heating rate (per unit volume):

$$\rho \frac{\delta q}{\delta t} = -\frac{dF}{dz} = -\int_{-1}^{+1} 2\pi \cdot \frac{dI}{dz} \cdot \mu \cdot d\mu \quad = \int_{-1}^{+1} 2\pi \cdot k_{ab} \cdot \frac{dI}{d\tau} \cdot \mu \cdot d\mu$$

$$= \int_{-1}^{+1} 2\pi \cdot k_{ab} \cdot (I - B) \cdot d\mu$$

$$= \int_{0}^{+1} 2\pi \cdot k_{ab} \cdot (I^{+} - B) \cdot d\mu + \int_{-1}^{0} 2\pi \cdot k_{ab} \cdot (I^{-} - B) \cdot d\mu$$

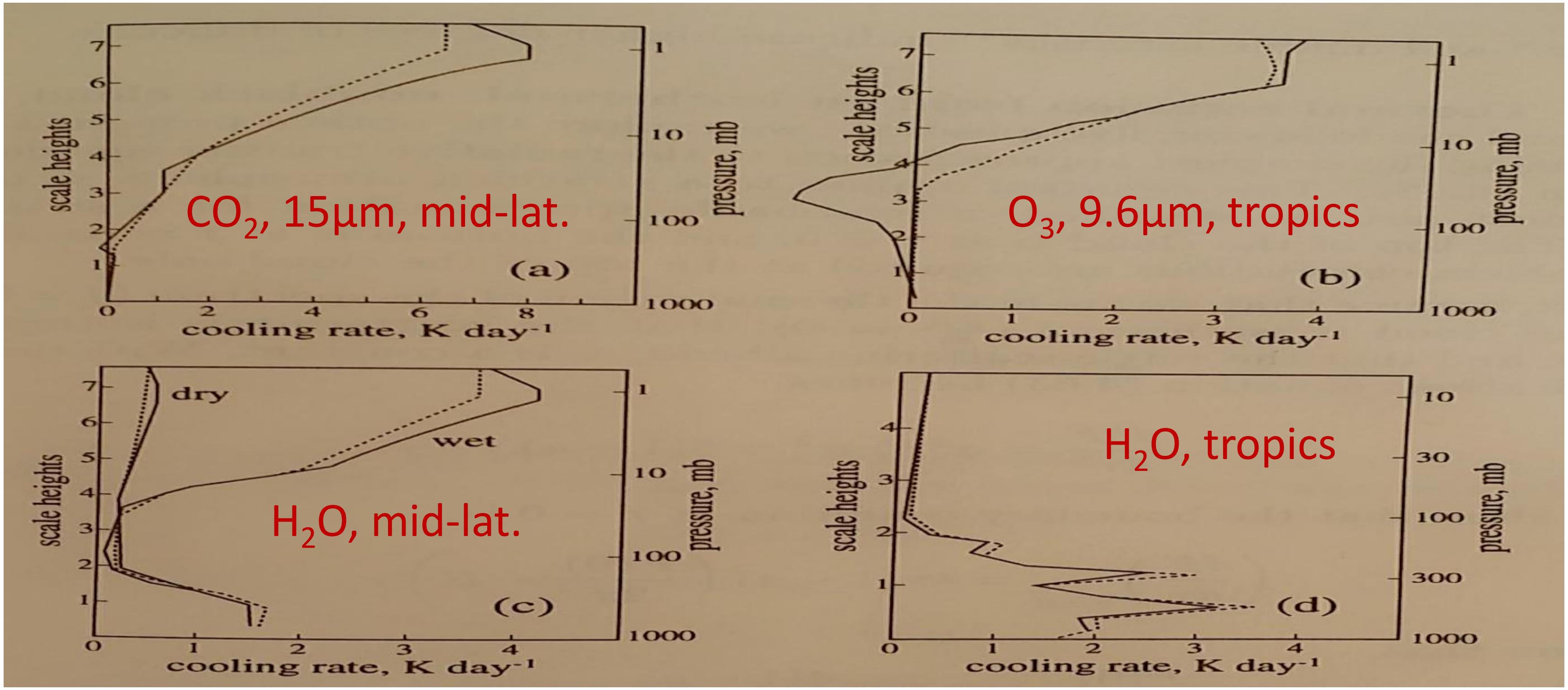
$$= -\int_{-1}^{0} 2\pi \cdot k_{ab} \cdot B \cdot e^{\tau/\mu} \cdot d\mu$$

$$= -2\pi \cdot k_{ab} \cdot B \cdot E_2(\tau)$$

Second exponential integral: $E_2(x) = \int_0^1 e^{-x/y} dy = \int_1^{\infty} \frac{e^{-xt}}{t^2} dt$

Radiance to Space Heating Rate

Cooling rate = – Heating rate



Scale height = 6-8.5 km; Solid line: exact; Dotted line: approximation

Richard Goody

Approximate Differential Equations for Diffuse Flux

- For many problems one concerns radiation flux in a stratified atmosphere only
- Direct solar radiance can be easily calculated with Bill's Law, thus not needed to be included in the radiative transfer equation

$$\mu \frac{dI}{d\tau} = I - J$$ (Diffuse radiance, stratified atmosphere)

$$J = \frac{\omega_0}{4\pi} \left\{ F_{\boxast} \cdot P(\tau, \theta, \mu, \theta_0, \varphi_0) + \int_{4\pi} I(\tau, \theta', \varphi') \cdot P(\tau, \theta, \varphi, \theta', \varphi') \cdot d\Omega' \right\} + (1 - \omega_0) B(\tau)$$

$$F_{\boxast} = F_0 e^{-\tau/\mu_0}$$

Method of Moments, Semi-Isotropic Field of Radiance

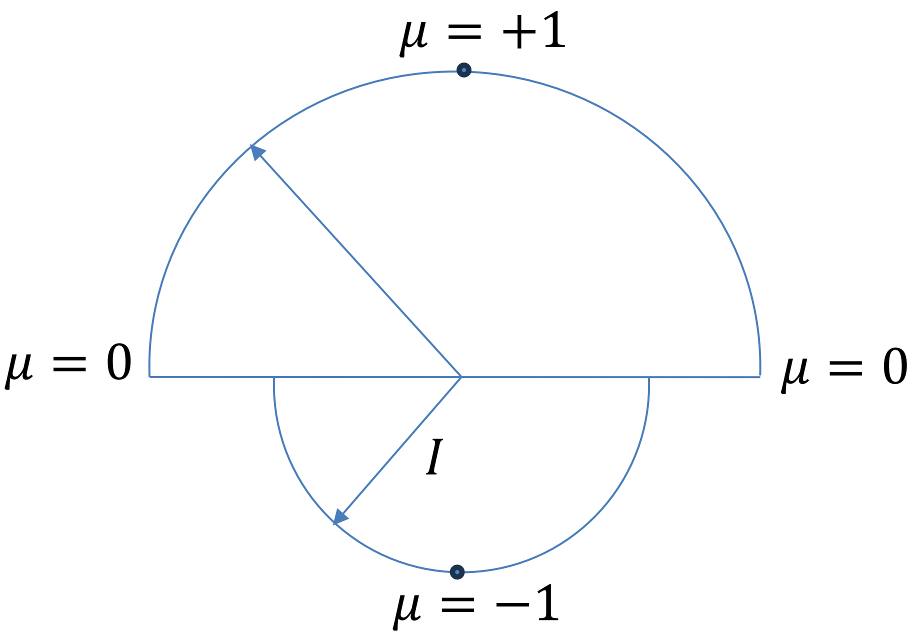
Method of moments 矩量法:

定义 $\int_{4\pi} I \cdot d\Omega = 4\pi\bar{I}$

定义 $\int_{4\pi} I \cdot \mu \cdot d\Omega = F$

近似 $\int_{4\pi} I \cdot \mu^2 \cdot d\Omega = \frac{4\pi}{3}\hat{I} \approx \frac{4\pi}{3}\bar{I}$

For semi-isotropic radiance:



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$\bar{I} = (I^+ + I^-)/2$

$F = \pi(I^+ - I^-)$ Upward: positive

Method of Moments, Semi-Isotropic Field of Radiance

We get:

$$\frac{dF}{d\tau} = 4\pi(1-\omega_0)(\bar{I} - B) - \omega_0 F_{\boxast} \quad \Leftarrow \quad 求 \int_{4\pi} \mu \frac{dI}{d\tau} d\Omega$$

$$\frac{4\pi}{3}\frac{d\bar{I}}{d\tau} = F(1-\omega_0 g) - \omega_0 g F_{\boxast}\mu_0 \quad \Leftarrow \quad 求 \int_{4\pi} \mu^2 \frac{dI}{d\tau} d\Omega$$

Here:

$$\frac{1}{4\pi}\int_{4\pi} P(\mu)\mu d\Omega = g$$

Asymmetric factor

$$\frac{1}{4\pi}\int_{4\pi} P(\mu,\varphi,\mu',\varphi')\mu d\Omega = g\mu'$$

For semi-isotropic air

Thus:

Diffuse flux

$$\frac{d^2F}{d\tau^2} = 3(1-\omega_0)(1-\omega_0 g)F - 4\pi(1-\omega_0)\frac{dB}{d\tau} - F_{\boxast}\left(\frac{\omega_0}{\mu_0} + 3\omega_0 g\mu_0(1-\omega_0)\right)$$

Radiative Transfer of Diffuse Flux

$$\frac{d^2F}{d\tau^2}$$
$$= 3(1-\omega_0)(1-\omega_0 g)F - 4\pi(1-\omega_0)\frac{dB}{d\tau}$$
$$- F_{⊛}\left(\frac{\omega_0}{\mu_0} + 3\omega_0 g\mu_0(1-\omega_0)\right)$$

Lower boundary conditions: $I^+ = B_g$ $F = 2\pi\left(B_g - \bar{I}\right)$

Upper boundary conditions: $I^- = 0$ $F = 2\pi\bar{I}$

思考题

Consider thermal emission from an isothermal, nonblack cloud. Suppose there is a stratus cloud with stratified atmosphere. The top of the cloud is at $\tau = 0$, and the bottom is at $\tau = \infty$. Assume that we are in the thermal region ($F_{\odot} = 0$), that the cloud is isothermal ($\frac{dB}{d\tau} = 0$), that the scattering is isotropic ($g = 0$), and that the single scattering albedo (ω_0) is a constant. Thus, the radiative transfer of diffuse flux becomes:

$$\frac{d^2F}{d\tau^2} = \alpha^2 F, \qquad \alpha^2 = 3(1-\omega_0)$$

(i) Show that the boundary condition at TOA ($\tau = 0$) is:

$$\left(\frac{dF}{d\tau}\right)_{\tau=0} = 4\pi(1-\omega_0)\left(\frac{F(0)}{2\pi} - B\right)$$

and:

$$\frac{F(0)}{\pi B} = \frac{4(1-\omega_0)}{2(1-\omega_0)+\sqrt{3(1-\omega_0)}}$$

(ii) The above formulae are approximate. What should be flux be for a black body ($\omega_0 = 0$)? Correct the above formula with a constant factor to allow for this error and calculate $\frac{F(0)}{\pi B}$ for $\omega_0 = 1.0, 0.8, 0.6, 0.4, 0.2, 0.0$.

Richard Goody