Equilibrium Climate

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思考题

Derive a relationship for the height of a given pressure surface (p) in terms of the pressure p_0 and temperature T_0 at sea level, assuming that the temperature T decreases uniformly with height at a rate of Γ (K km^{-1}).

$$z = \frac{T_0}{\Gamma}\left[1 - \left(\frac{p}{p_0}\right)^{R\Gamma/g}\right]$$

高度计

This is the basis for the calibration of aircraft altimeters

思考题

Lifting all assumptions for air parcel, except that the environment is still in hydrostatic equilibrium.

(a) Show that when a parcel of dry air at temperature T' moves adiabatically in ambient air with temperature T, the temperature lapse rate of the air parcel is given by

$$\Gamma = -\frac{dT'}{dz} = \frac{T'}{T}\frac{g}{c_p}$$

(b) Explain why the lapse rate in this case differs from the dry adiabatic lapse rate $\Gamma_d = \frac{g}{c_p}$

思考题

- Assuming the truth of the second law of thermodynamics, prove that an isolated ideal gas can expand spontaneously (e.g., into a vacuum) but cannot contract spontaneously

- One kilogram of ice at $0^{o}C$ is placed in an isolated container with 1 kg of water at $10^{o}C$ and 1 atm.
  (a) How much of the ice melts?
  (b) What change is there in the entropy of the universe due to the melting of the ice? (specific heat of water is 4218 J K^{-1} kg^{-1})

思考题

By differentiating the enthalpy function ($h = u + p\alpha$), show that

$$\left(\frac{\partial p}{\partial T}\right)_s = \left(\frac{\partial s}{\partial \alpha}\right)_p$$

where s is entropy.

Note：

$$dh = Tds + \alpha dp$$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial y}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial y}{\partial x_j}\right)$$

Show that this is equivalent to the Clausius-Clapeyron Equation.

This is one of the Maxwell's four thermodynamic equations.

Radiative Equilibrium Temperature: Vertical Structure

Radiative equilibrium for stratified atmosphere:

$$\rho \frac{\delta q}{\delta t} = -\frac{\partial F}{\partial z} = 0$$

$$F = F_T - F_S = constant = 0$$

Assuming no extinction of solar radiation by atmosphere

So, solar radiation flux (downward is negative):

$$F_S = x \cdot S \cdot (1 - R) = constant$$

$$x = \frac{\pi r^2}{4\pi r^2} = 1/4 \quad \text{for global mean annual mean}$$

$$x \approx 1/\pi \quad \text{for tropical mean annual mean}$$

Thus: $F_T = constant$

Radiative Equilibrium Temperature: Vertical Structure

For radiative transfer of thermal radiation flux for

stratified atmosphere with no scattering:

$$\frac{dF_T}{d\tau} = 4\pi(1-\omega_{0\_T})\left(\bar{I} - B\right) = 0$$

$$\frac{d^2F_T}{d\tau^2} = 3(1-\omega_{0\_T})(1-\omega_{0\_T}g)F_T - 4\pi(1-\omega_{0_T})\frac{dB}{d\tau} = 0$$

$$\omega_{0\_T} = 0$$

We get: $\bar{I} = B$

$$4\pi \cdot \frac{dB}{d\tau} = 3F_T$$

$$B(\tau) - B(0) = \frac{3F_T\tau}{4\pi}$$

Radiative Equilibrium Temperature: Vertical Structure

Solution:

$$B = \bar{I}$$

$$B(\tau) - B(0) = \frac{3F_T\tau}{4\pi}$$

Boundary conditions:

$$B_g - B(\tau_1) = \frac{F_T}{2\pi}$$

$$B(0) = \frac{F_T}{2\pi}$$

Thus:

$$B(\tau) = \frac{\sigma}{\pi}T(\tau)^4 = \frac{F_T}{2\pi}\left(1 + \frac{3\tau}{2}\right) \qquad T(\tau) = \sqrt[4]{\frac{F_T}{2\sigma}\left(1 + \frac{3\tau}{2}\right)}$$

$$B(\tau_1) = \frac{\sigma}{\pi}T(\tau_1)^4 = \frac{F_T}{2\pi}\left(1 + \frac{3\tau_1}{2}\right) \Rightarrow T(\tau_1) = \sqrt[4]{\frac{F_T}{2\sigma}\left(1 + \frac{3\tau_1}{2}\right)}$$

$$B_g = \frac{\sigma}{\pi}{T_g}^4 = \frac{F_T}{2\pi}\left(2 + \frac{3\tau_1}{2}\right) \qquad T_g = \sqrt[4]{\frac{F_T}{2\sigma}\left(2 + \frac{3\tau_1}{2}\right)}$$

Temperature discontinuity at τ_1

Radiative Equilibrium Temperature: Vertical Structure

$$T(\tau) = \sqrt[4]{\frac{F_T}{2\sigma}\left(1 + \frac{3\tau}{2}\right)} \qquad T(0) = \sqrt[4]{\frac{F_T}{2\sigma}} \qquad T(\tau_1) = \sqrt[4]{\frac{F_T}{2\sigma}\left(1 + \frac{3\tau_1}{2}\right)} \qquad T_g = \sqrt[4]{\frac{F_T}{2\sigma}\left(2 + \frac{3\tau_1}{2}\right)}$$

In many cases (e.g., CO_2):

Vol. mixing ratio $\quad r \approx constant$

Number conc. $\quad n(z) \approx r \cdot n_{air}(z) \approx n_1 \cdot e^{(-z/H)}$

H = scale height

Thus: $\quad k(z) \approx n(z) \cdot \sigma = k_1 \cdot e^{(-z/H)}$

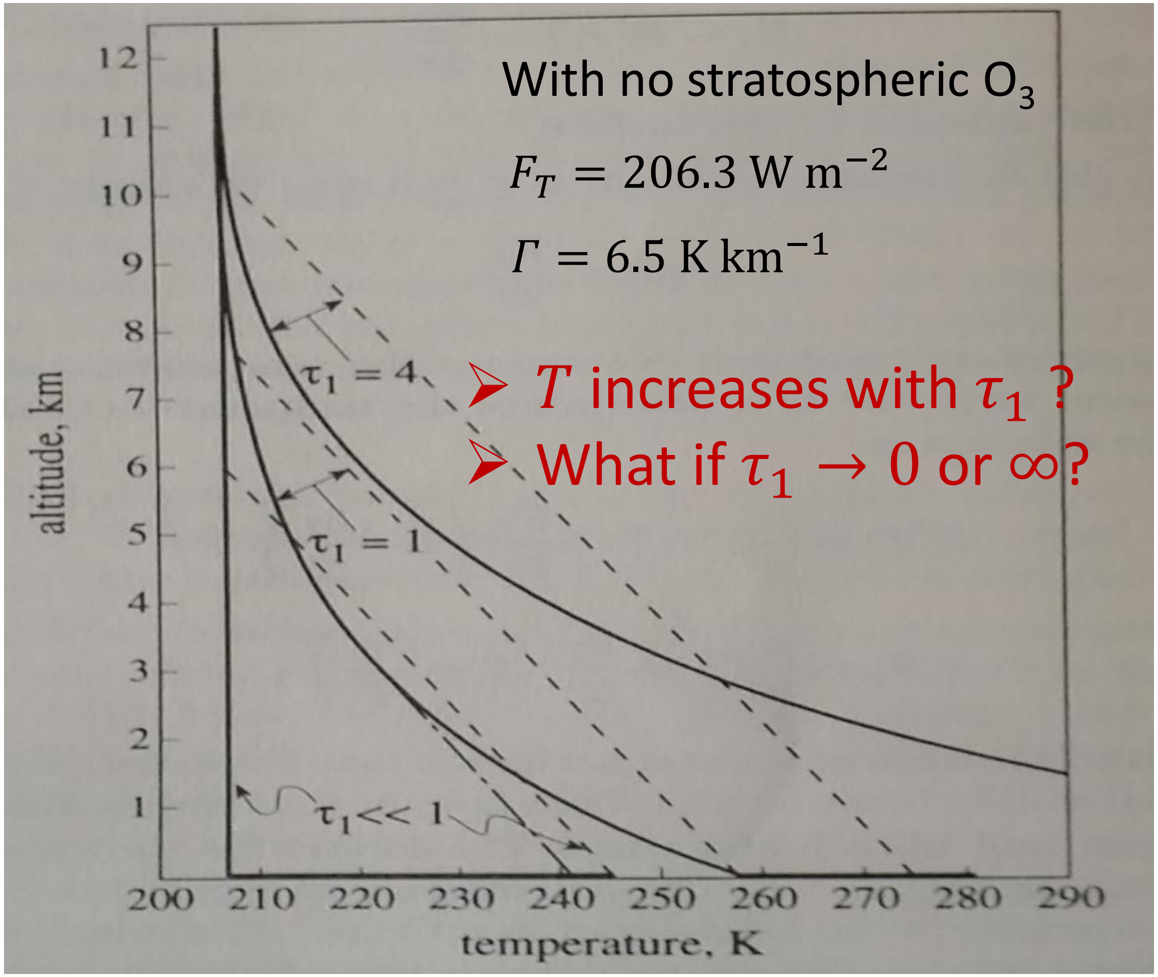
σ = absorption cross section

$$\tau(z) = \int_z^\infty k(z) \cdot dz = k_1 H e^{(-z/H)} = \tau_1 \cdot e^{(-z/H)}$$

Temperature Discontinuity

$$\tau(z) = \tau_1 \cdot e^{(-z/H)}$$

Thus:



- T increases with τ_1 ?
- What if $\tau_1 \rightarrow 0$ or ∞?

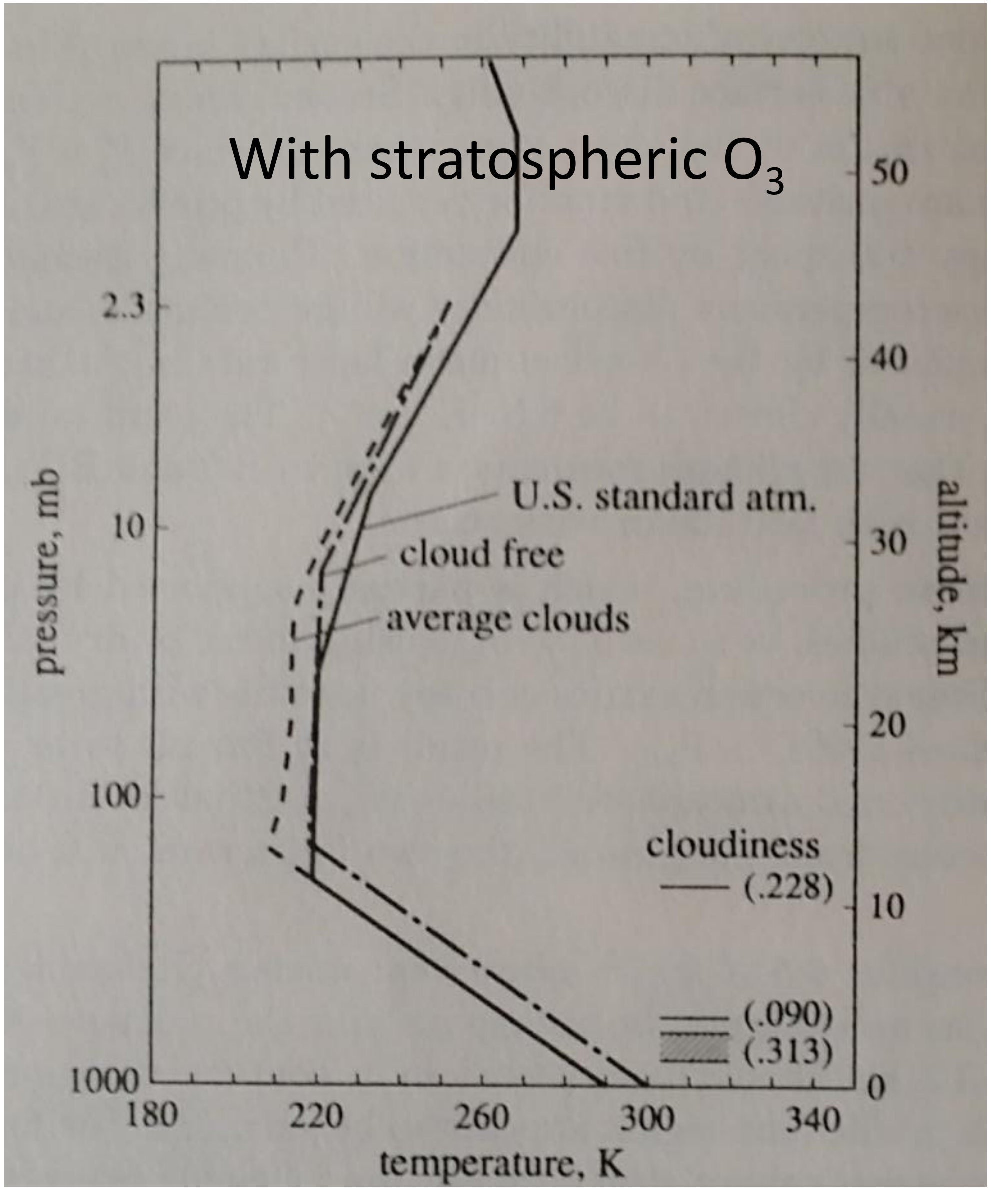
Radiation-Convection in Troposphere

$F_{s\_cloudfree} = 299 \text{ W m}^{-2}$

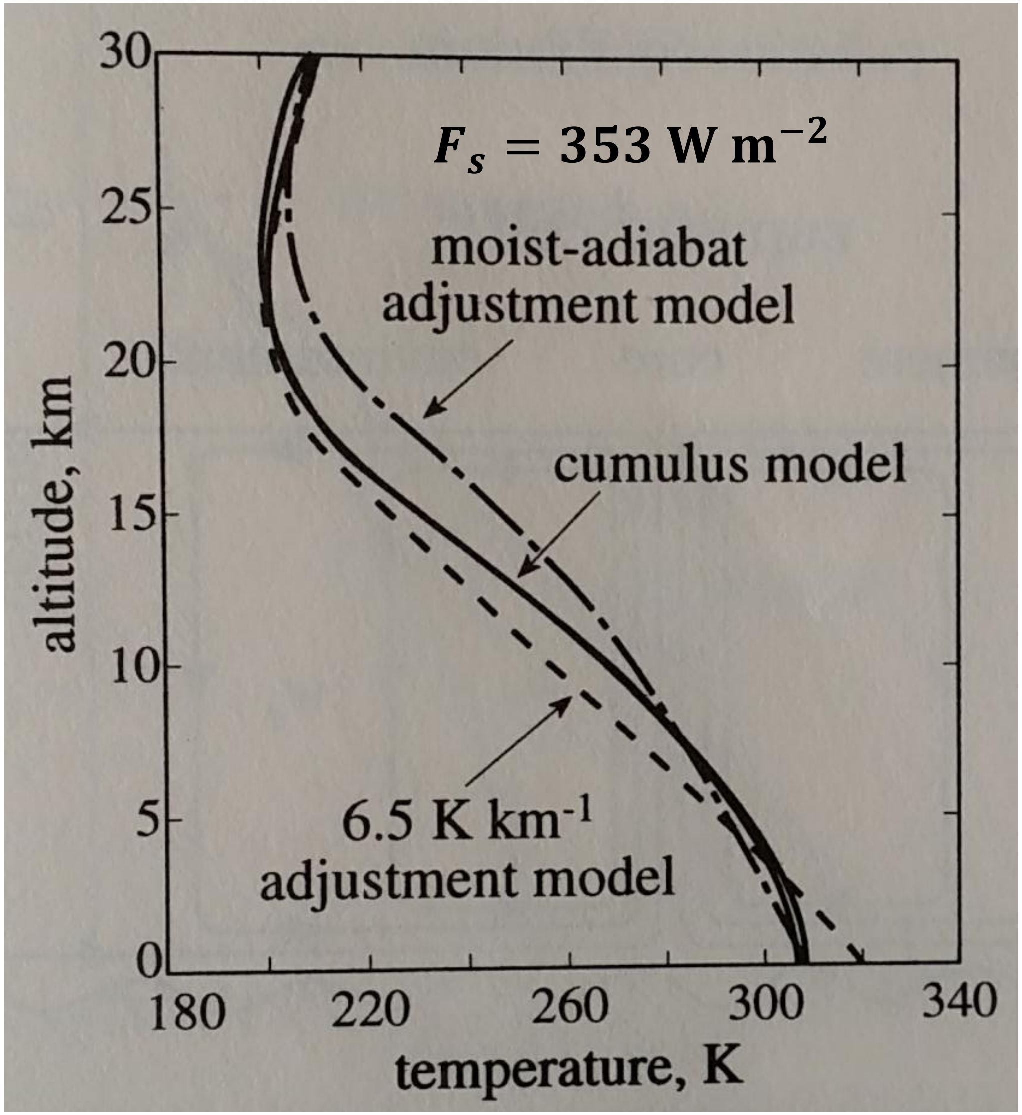
$F_{s\_cloudy} = 228 \text{ W m}^{-2}$

$\Gamma = 6.5 \text{ K km}^{-1}$

$\text{SZA} = 60^{\circ}$



Radiation-Convection in Tropical Troposphere



The Emission Level

At the emission level τ_e,

black body emission is equal to planetary emission

$$F_T(\tau = 0) = \pi B_e = \sigma {T_e}^4$$

$$B_e = B(\tau_e) = \frac{F_T}{2\pi}\left(1 + \frac{3\tau_e}{2}\right) = \frac{F_T}{\pi}$$

Therefore:

$$\tau_e = \frac{2}{3}$$

The Chapman Layer

From the thermal radiance equation (assuming $\tau_1 >> 1$):

[Ignoring the thermal radiation window at ~ 8-12 μm]

$$I_T^+(\tau = 0, \mu) = B(T_g) \cdot e^{-\tau_1/\mu} + \frac{1}{\mu} \cdot \int_0^{\tau_1} B(T(\tau)) \cdot e^{-\tau/\mu} \cdot d\tau$$

at TOA

$$\approx \frac{1}{\mu} \cdot \int_0^{\tau_1} B(T(\tau)) \cdot e^{-\tau/\mu} \cdot d\tau$$

$$= \int_0^{\infty} B(z) \frac{-1}{\mu} \cdot \frac{d\tau}{dz} \cdot e^{-\tau/\mu} \cdot dz \quad = \int_0^{\infty} B(z) \cdot h(z, \mu) \cdot dz$$

Where:

$$h(z, \mu) = \frac{-1}{\mu} \cdot \frac{d\tau}{dz} \cdot e^{-\tau/\mu}$$

h = Kernel function: $\int_0^{\infty} h(z, \mu) dz = 1$

$$= \frac{1}{H\mu} \cdot \tau(z) \cdot e^{-\tau(z)/\mu}$$

[for $\tau(z) = \tau_1 \cdot e^{-z/H}$]

At maximum h : $h(max) = \frac{1}{eH}$ $\quad$ $\tau(max) = \mu$

The Chapman Layer

Re-arrange the kernel function:

$$\frac{h(z)}{h(max)} = \frac{\tau(z)}{\tau(max)} e^{1-\frac{\tau(z)}{\tau(max)}}$$

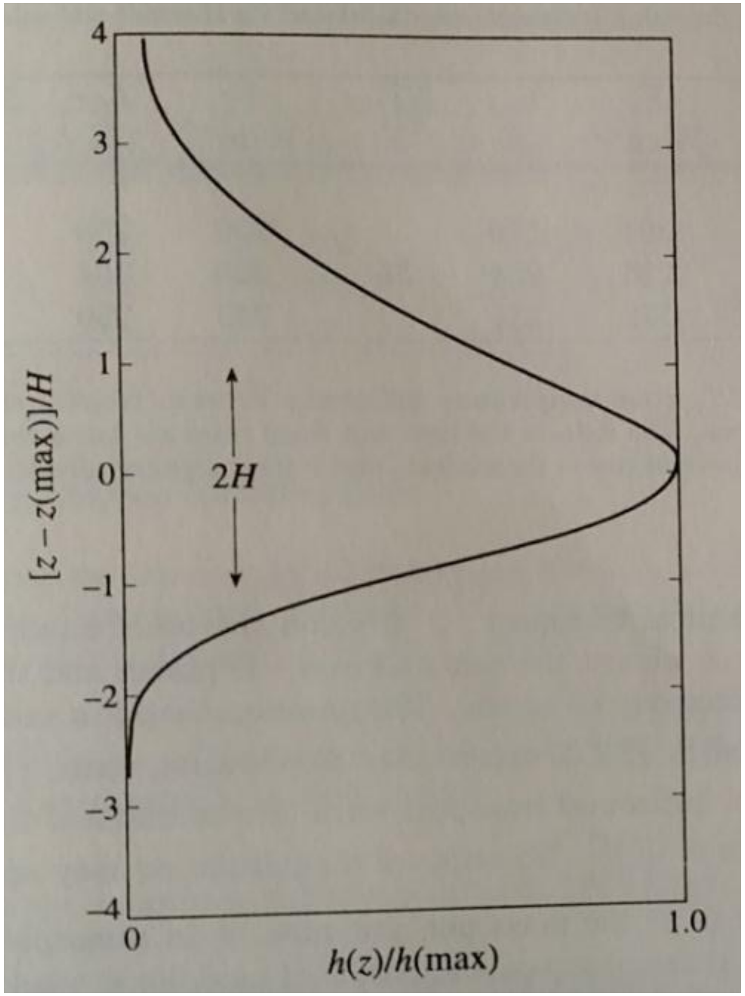
Where:

$$h(max) = \frac{1}{eH}$$

$$\tau(max) = \mu$$

$$z(max) = H \cdot ln\left(\frac{\tau_1}{\mu}\right)$$

$$\frac{\tau(z)}{\tau(max)} = e^{-\frac{z-z(max)}{H}}$$



The Chapman Layer

Mean value theorem for thermal radiation flux:

$$F_T^+(z) = \int_0^{2\pi} \int_0^1 I_T^+(z,\mu) \cdot \mu \cdot d\mu \cdot d\varphi$$

$$= 2\pi \cdot \int_0^1 I_T^+(z,\mu) \cdot \mu \cdot d\mu$$

$$= \pi \cdot I_T^+(z, \hat{\mu})$$

Many investigators found that globally, $\hat{\mu} = \frac{2}{3}$

Therefore, integrating over all directions (μ),
kernel function for thermal flux to space peaks at the emission level

[Ignoring the thermal radiation window at ~ 8-12 µm]

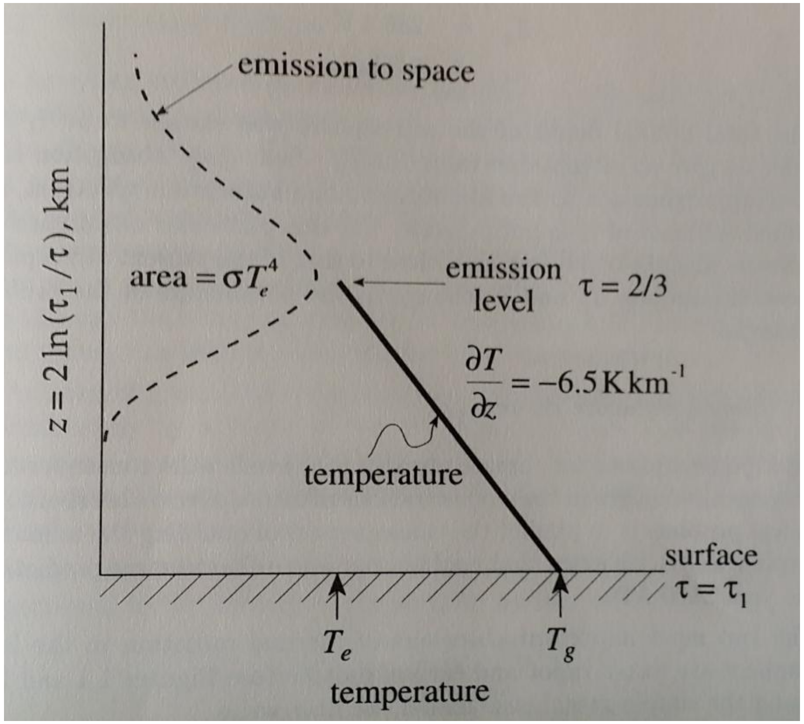
Semi-gray Greenhouse Model

Semi-gray model:

$$\tau_e = \tau_1 e^{-z_e/H} = 2/3$$

➡ $$z_e = H \ln \frac{3\tau_1}{2}$$

➡ $$T_g = T_e + \Gamma_{obs} H \ln \frac{3\tau_1}{2}$$ in trop.



Given that:

$$F_S = 236.6 \ \mathrm{W\,m^{-2}}$$

$$\Gamma_{obs} = 6.5 \ \mathrm{K\,km^{-1}}$$

For H_2O: $H = 2$ km

$$\tau_1 = 8$$

Ignoring thermal rad. window at ~ 8-12 μm

Therefore:

$$Z_e = 5.0 \ \mathrm{km}$$

$$T_e = 254.1 \ \mathrm{K}$$

$$T_g = 286.4 \ \mathrm{K}$$

Climate Sensitivity with No Feedbacks other than Planck's

$$\frac{\partial T_g}{\partial \ln \tau_{1,CO_2}} = \Gamma_{obs} H \frac{\partial \ln \tau_1}{\partial \ln \tau_{1,CO_2}} = \Gamma_{obs} H \frac{\tau_{1,CO_2}}{\tau_1}$$

Current atmosphere, CO_2 and H_2O only:

$$\tau_1 = \tau_{1,CO_2} + \tau_{1,H_2O}$$

$$\frac{\tau_{1,CO_2}}{\tau_1} \approx 0.125$$

Climate sensitivity is small:

$$\frac{\partial T_g}{\partial \ln \tau_{1,CO_2}} = 1.6\ K$$

$\boldsymbol{\Delta T_g = 1.1\ K}$ for 2 x $[CO_2]$

Water Vapor Feedback

Clausius-Clapeyron Equation

At constant air pressure

$$\frac{1}{w_s}\frac{dw_s}{dT} \approx \frac{1}{e_s}\frac{de_s}{dT} = \frac{L_v}{R_v T^2}$$

Equilibrium (saturation) water vapor mixing ratio near the surface will increase by 7.2% for every 1 K increase in temperature around 273 K, that is,

w_s will double for every 10 K increase in T_g for $T_{g,0} = 273\ \mathrm{K}$

$$w_s = w_{s,0} \cdot e^{\frac{L_v}{R_v}\cdot\frac{T_g - T_{g,0}}{T_g \cdot T_{g,0}}} \approx w_{s,0} \cdot e^{\frac{L_v}{R_v}\cdot\frac{T_g - T_{g,0}}{(T_{g,0}+10)\cdot T_{g,0}}} \approx w_{s,0} \cdot e^{\frac{T_g - T_{g,0}}{10}\ln 2}$$

Climate Sensitivity with Water Vapor & Planck's Feedbacks

If surface RH and vertical profile of water vapor are unchanged:

$$\tau_{1,H_2O} = \tau_{1,H_2O,0} \cdot e^{\frac{T_g - T_{g,0}}{10} \ln 2}$$

We get:

$$\frac{\partial T_g}{\partial \ln \tau_{1,CO_2}} = \alpha \cdot \Gamma_{obs} H \frac{\tau_{1,CO_2}}{\tau_1}$$

Climate sensitivity:

$\boldsymbol{\Delta T_g = 5.2\ K}$ for 2 x [CO_2]

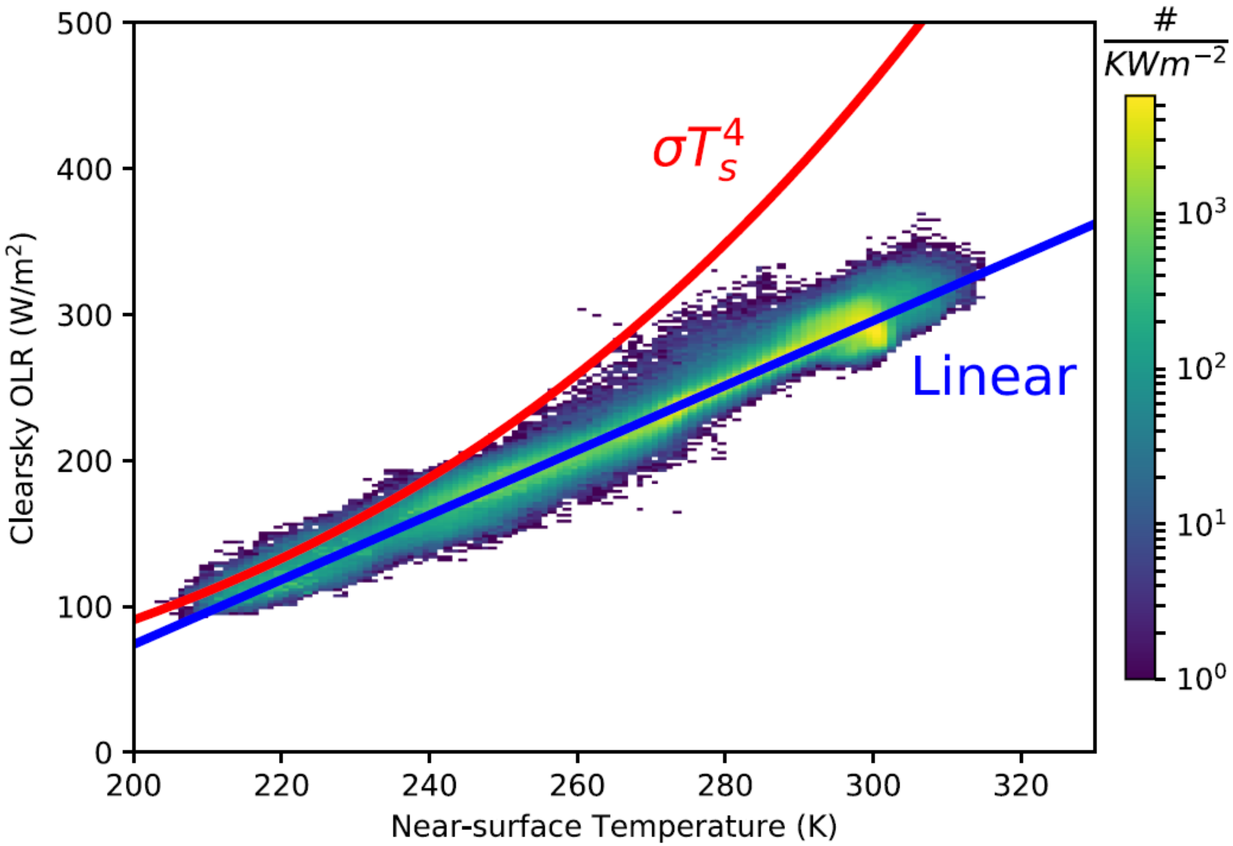
- H_2O feedback greatly enhances climate sensitivity
- IPCC AR6 estimate: 2.5-4.0 K (likely range)
- Runaway greenhouse effect?

Where:

$$\alpha = \left(1 - \Gamma_{obs} H \frac{1}{\tau_1} \frac{\partial \tau_{1,H_2O}}{\partial T_g}\right)^{-1} = \left(1 - \Gamma_{obs} H \frac{\tau_{1,H_2O}}{\tau_1} \frac{\ln 2}{10}\right)^{-1} = 4.7$$

Linking OLR and Near-Surface Temperature

$$T_e = T_g - \Gamma_{obs} H \ln \frac{3\tau_1}{2}$$



Koll et al., 2018, PNAS

Effects of Clouds

- Low clouds (≤ 4 km): changes albedo
- High clouds (~ 8 km): changes albedo and emission layer height

	R	z_e, km	T_g, K	ΔT_g, K
No could	0.12	5	302	
Complete high clouds	0.3	8	307	+5
Complete low clouds	0.7	5	234	-68

Energy Balance Climate Models: Meridional Change

$$c\frac{\partial T_g(x)}{\partial t} = solar\ term + thermal\ term + dynamic\ term$$

$$solar\ term = \frac{S}{4} \cdot s(x) \cdot [1 - A(x)]$$

$$s(x) = 1 - 0.241\left(3x^2 - 1\right)$$

$$thermal\ term = -\left[211.1 + 1.55T_g(x)\right]$$

$$dynamical\ term = \frac{\partial}{\partial x}\left[\left(1 - x^2\right) \cdot D \cdot \frac{\partial T_g(x)}{\partial x} \cdot s(x)\right]$$

c = specific heat cap.

S = solar irradiance

x = sin (latitude)

D = diffusion coefficient

A = surface albedo

T_g = ground temp. in ^{o}C

Energy Balance Climate Models

Albedo:

- Ice or snow: $A_i = 0.6$, $T_g \leq -10^oC$
- Ocean or land surface: $A_s = 0.3$, $T_g > -10^oC$

x_i = ice/snow line (where $T_g = -10\ ^0C$)

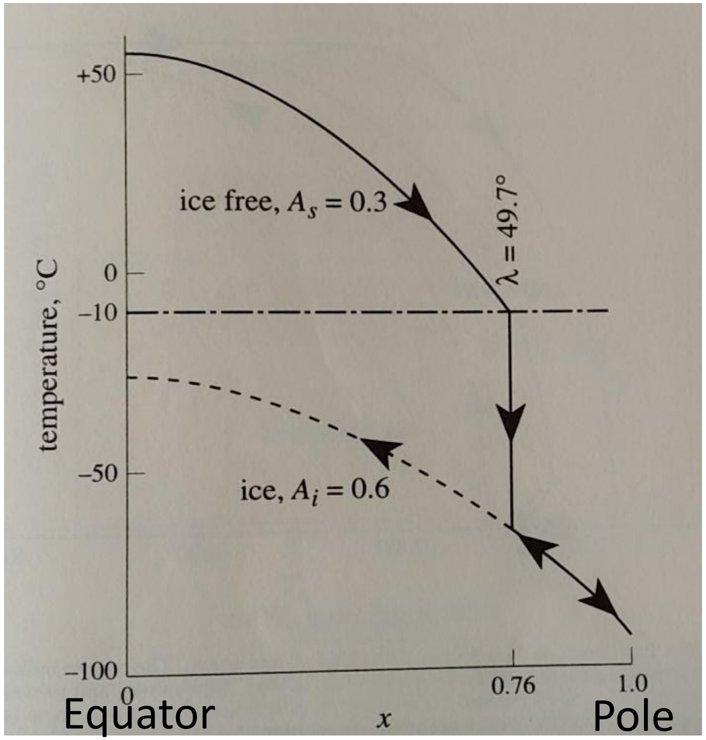
For steady state ($\frac{\partial T_g}{\partial t} = 0$), and no dynamical term (D = 0):

$$T_g(x) = \frac{(1-A)\cdot S}{4\times 1.55}\left[1-0.241\left(3x^2-1\right)\right] - \frac{211.1}{1.55}$$

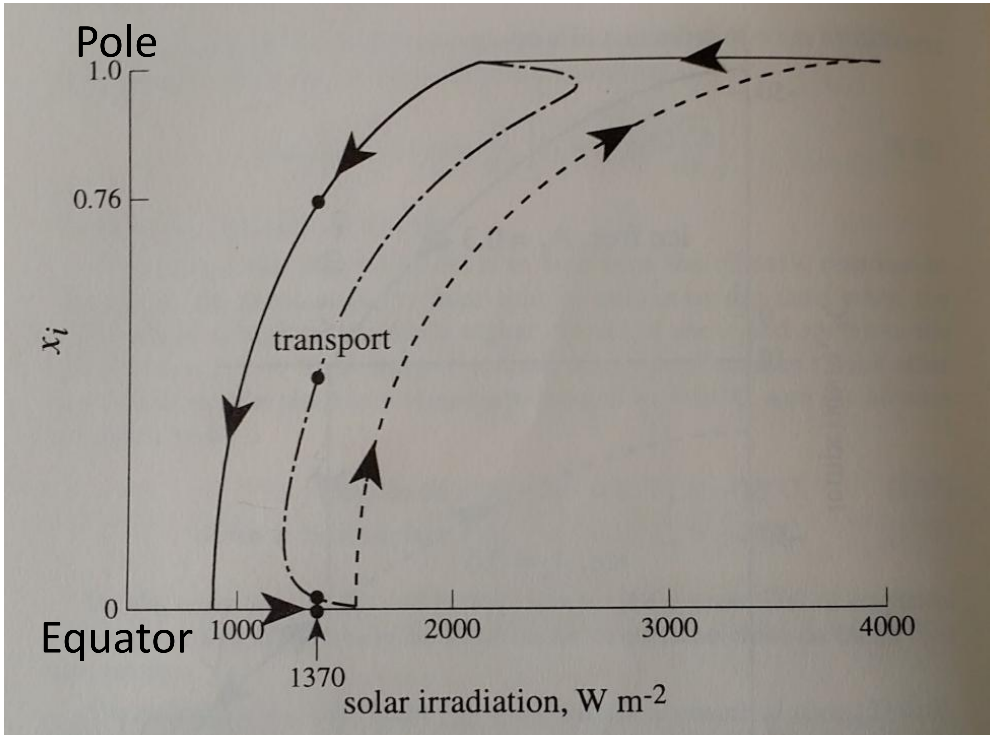
$$S = \frac{(211.1 - 10\times 1.55)\times 4}{(1-A)\cdot\left[1-0.241\left(3x_i^2-1\right)\right]}$$

Energy Balance Climate Models

Temperature as a function of latitude



Position of the ice line as a function of insolation



Hysteresis 迟滞现象:

The dependence of the output of a system not only on its current input, but also on its history of past inputs

思考题

- ✓ For a planet similar to the Earth but with no water, how would its surface temperature and air temperature be like? Consider a two-box model under energy balance.
- ✓ Why is there a lapse rate feedback? What are the causes of the meridional dependence of this feedback?
- ✓ How would surface temperature, air temperature and water vapor change if anthropogenic greenhouse gas concentrations continue to increase? Runaway greenhouse effect?